

Relativistic effects and primordial non-Gaussianity in the matter density fluctuation

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We present the third-order analytic solution of the matter density fluctuation in the proper-time hypersurface of nonrelativistic matter flows by solving the nonlinear general relativistic equations. The proper-time hypersurface provides a coordinate system that a local observer can set up without knowledge beyond its neighborhood, along with physical connections to the local Newtonian descriptions in the relativistic context. The initial condition of our analytic solution is set up by the curvature perturbation in the comoving gauge, clarifying its impact on the nonlinear evolution. We compute the effective non-Gaussian parameters due to the nonlinearity in the relativistic equations. With proper coordinate rescaling, we show that the equivalence principle is respected and the relativistic effect vanishes in the large-scale limit.

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Rapid developments in large-scale galaxy surveys over the past decades have enabled the precision measurements of galaxy clustering, which can be used to probe the nature of dark energy and the perturbation generation mechanism in the early Universe [1]. In parallel, the recent theoretical development ([2–6]; see [7] for review) has revealed that the subtle relativistic effects are present in galaxy clustering, providing new opportunities to extract additional and critical information about the gravity on large scales and the initial conditions for structure formation. In particular, the relativistic formalism has been extended [8–10] to the second-order in perturbations for the computation of higher-order statistics such as the bispectrum.

One of the critical elements in the relativistic formalism is galaxy bias, which relates the galaxy number density to the underlying matter distribution. Beyond the linear order in perturbations, however, galaxy bias poses a nontrivial problem due to the gauge issues in general relativity. It was shown [11] that the proper-time hypersurface of nonrelativistic matter flows provides a physical description of the local observer, moving with dark matter and baryons that will collapse to form galaxies. This physical justification has led us to study the nonlinear relativistic effects of the matter density fluctuation in the proper-time hypersurface of nonrelativistic matter flows.

This is timely and interesting, for there has been recent conflict in literature — It is argued in [12, 13] by performing the second-order relativistic calculations that the nonlinear evolution of gravity generates the local-type non-Gaussianity, which would in turn exhibit a prominent signature in galaxy clustering on large scales. On the other hands, in [14–16] the opposite claim is asserted that the observable quantities are *not* affected by these nonlinear relativistic effects of gravity, while the calculations are in general based on studying the special case (or the squeezed limit), where only the linear-

order calculations are required. In this article, we present the third-order relativistic calculations of the matter density fluctuation in the proper-time hypersurface, providing the essential tool for computing the relativistic effects in the higher-order galaxy clustering statistics and explicitly resolving the hotly debated issues of the nonlinear relativistic effects in general relativity. Throughout the article we use a, b, c, \dots for the spacetime indices and i, j, k, \dots for the spatial ones.

Let us consider a Friedmann-Robertson-Walker universe with an irrotational pressureless medium of nonrelativistic matter, encompassing baryons and dark matter on large scales. A local observer moving with this nonrelativistic matter flow is described by its four velocity u^a , and the energy-momentum tensor in this case is greatly simplified as $T_{ab} = \rho_m u_a u_b$, where ρ_m is the energy density of the fluid measured by the local observer. As our temporal gauge condition, we choose the *comoving gauge*, where local observers see vanishing energy flux $T^0_i = 0$. In this case, the global time coordinate is synchronized with the proper-time, and the local observer u^a is aligned with the geometric normal observer n^a ($n_i = 0$) [11], so that the local observer moves along the geodesic $N = 1$, where N is the lapse function in the Arnowitt-Deser-Misner (ADM) formalism [17, 18].

Given that the local observer coincides with the normal observer in our temporal gauge condition, the expansion θ and the shear σ_{ij} of the flow are related to the extrinsic curvature tensor K_{ij} of the 3-hypersurface

$$K_{ij} \equiv \frac{1}{2N} (N_{i;j} + N_{j;i} - \dot{h}_{ij}) = -u_{i;j}, \quad (1)$$

as

$$-\theta = K = h^{ij} K_{ij}, \quad \text{and} \quad -\sigma_{ij} = K_{ij} - \frac{1}{3} h_{ij} K, \quad (2)$$

where N_i is the ADM shift vector, the dot is the time derivative, and the colon is the covariant derivative with respect to the projection tensor $h_{ab} \equiv g_{ab} + u_a u_b$, which is the induced spatial metric of 3-hypersurface.

Moreover, as our spatial gauge condition which matters beyond the linear order in perturbations, we choose the spatial

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C-gauge condition [19] to have only the diagonal part in the spatial metric

$$h_{ij} \equiv a^2(1 + 2\varphi)\bar{g}_{ij}, \quad (3)$$

with curvature perturbation φ and the scale factor a . This results in non-vanishing shift vector $g_{0i} = N_i \equiv -\nabla_i \chi$, where \bar{g}_{ij} is the background 3-metric and the covariant derivative ∇_i is with respect to \bar{g}_{ij} . Notice that the spatial B-gauge condition combined with our temporal comoving gauge condition, the so-called comoving-synchronous gauge, yields the simplest metric

$$h_{ij} \equiv a^2[(1 + 2\varphi)\bar{g}_{ij} + 2\nabla_i \nabla_j \gamma] \quad \text{and} \quad g_{0i} = 0, \quad (4)$$

at the cost of nonvanishing off-diagonal term γ in the spatial metric. Our choice of gauge conditions corresponds to a global coordinate system with the proper-time of a local observer, and this choice leaves no residual gauge mode [11]. Note that we assume no vector or tensor perturbation present in the initial condition.

With our temporal and spatial gauge conditions, the continuity, the Raychaudhuri, the ADM energy and momentum

constraint equations become the nonlinear dynamical equations for the density perturbation $\delta \equiv \rho/\bar{\rho}_m - 1$ and the extrinsic curvature perturbation $\kappa \equiv 3H + K$ as

$$\dot{\delta} - \kappa = N^i \nabla_i \delta + \kappa \delta, \quad (5)$$

$$\dot{\kappa} + 2H\kappa - 4\pi G\bar{\rho}_m \delta = N^i \nabla_i \kappa + \frac{1}{3}\kappa^2 + \sigma^{ab}\sigma_{ab}, \quad (6)$$

$$\delta R = \sigma^{ab}\sigma_{ab} + 4H\kappa - \frac{2}{3}\kappa^2 + 16\pi G\bar{\rho}_m \delta, \quad (7)$$

$$\sigma_{i;j}^j = \frac{2}{3} \nabla_i \kappa, \quad (8)$$

where δR is the perturbation of the intrinsic curvature of the 3-hypersurface and $H = \dot{a}/a$ is the Hubble parameter. To the third order in perturbations, each perturbation component can be readily computed in terms of metric perturbations [18, 20].

The ADM momentum constraint (8) and extrinsic curvature (2) provide auxiliary equations for the comoving gauge curvature φ and the scalar shear χ to be solved for δ and κ , and they are explicitly

$$\begin{aligned} \kappa + \frac{\Delta}{a^2} \chi &= \frac{1}{a^2} \left[2\varphi \Delta \chi (1 - 2\varphi) - \nabla^i \varphi \nabla_i \chi (1 - 4\varphi) + \frac{3}{2} \Delta^{-1} \nabla^i (\nabla_i \chi \Delta \varphi + \nabla_j \nabla_i \varphi \nabla^j \chi) \right] \\ &\quad - \frac{3}{a^2} \Delta^{-1} \nabla^i \left[2\varphi (\nabla^j \chi \nabla_j \nabla_i \varphi + \Delta \varphi \nabla_i \chi) + \frac{1}{2} (\nabla_i \chi \nabla_j \varphi + 3 \nabla_j \chi \nabla_i \varphi) \nabla^j \varphi \right], \end{aligned} \quad (9)$$

$$\kappa + 3\dot{\varphi} + \frac{\Delta}{a^2} \chi = 6\varphi \dot{\varphi} + \frac{1}{a^2} [2\varphi \Delta \chi (1 - 2\varphi) - \nabla_i \chi \nabla^i \varphi (1 - 4\varphi)], \quad (10)$$

where Δ^{-1} is the inverse Laplacian operator. Combining these two equations with the geodesic condition $N = 1$, we obtain the evolution equation for the curvature perturbation

$$\dot{\varphi} = 2\varphi \dot{\varphi} - \frac{1}{2a^2} \Delta^{-1} \nabla^i \left[\nabla^j \chi \nabla_j \nabla_i \varphi + \Delta \varphi \nabla_i \chi - 4\varphi (\nabla^j \chi \nabla_j \nabla_i \varphi + \Delta \varphi \nabla_i \chi) - (\nabla_i \chi \nabla_j \varphi + 3 \nabla_j \chi \nabla_i \varphi) \nabla^j \varphi \right], \quad (11)$$

then we arrive at the well-known result that the comoving gauge curvature is conserved at the linear order in perturbations: $\varphi^{(1)} \equiv \mathcal{R}^{(1)}(\mathbf{x})$, where \mathcal{R} is the initial condition and the superscript (n) means n -th order in perturbation. To simplify the time evolution, we now assume that the universe is matter-dominated (In a Λ CDM universe, the time-dependence of the solution is more complicated. However, the spatial function in Eq. (18) is identical, leaving our conclusion on the effective non-Gaussian parameters unaffected.) Equation (11) can then be analytically integrated at each order in perturbations, and up to third order the solution is found as

$$\begin{aligned} \varphi(t, \mathbf{x}) &= \mathcal{R} + 2\mathcal{R}\varphi^{(2)} - \frac{2}{5(aH)^2} \left\{ \frac{1}{4} \nabla_i \mathcal{R} \nabla^i \mathcal{R} + \frac{1}{2} \Delta^{-1} \nabla^i \left[\Delta \mathcal{R} \nabla_i \mathcal{R} + \nabla_j \nabla_i \mathcal{R} \nabla^j \left(\frac{5}{2} H \chi_1^{(2)} \right) + \nabla_i \left(\frac{5}{2} H \chi_1^{(2)} \right) \Delta \mathcal{R} \right] \right. \\ &\quad \left. - 2\Delta^{-1} \left[(\nabla^i \nabla^j \mathcal{R} \nabla_i \nabla_j \mathcal{R} + \Delta \mathcal{R} \Delta \mathcal{R} + 2\nabla^i \mathcal{R} \Delta \nabla_i \mathcal{R}) \mathcal{R} + 3\nabla_i \nabla_j \mathcal{R} \nabla^i \mathcal{R} \nabla^j \mathcal{R} + 2\Delta \mathcal{R} \nabla^i \mathcal{R} \nabla_i \mathcal{R} \right] \right\} \\ &\quad + \frac{1}{4} \Delta^{-1} \nabla^i \left(\nabla_j \nabla_i \mathcal{R} \Delta^{-1} \nabla^j \frac{\kappa_2^{(2)}}{H} + \Delta^{-1} \nabla_i \frac{\kappa_2^{(2)}}{H} \Delta \mathcal{R} \right) - \frac{1}{10(aH)^2} \Delta^{-1} \nabla^i \left(\nabla_j \nabla_i \varphi^{(2)} \nabla^j \mathcal{R} + \nabla_i \mathcal{R} \Delta \varphi^{(2)} \right), \end{aligned} \quad (12)$$

with the second-order scalar shear $5H\chi_1^{(2)}/2 = \Delta^{-1}(\mathcal{R}\Delta\mathcal{R})/2 - 3\Delta^{-2}\nabla_i\nabla_j(\mathcal{R}\nabla^i\nabla^j\mathcal{R})$ in Eq. (9) and $\kappa_2^{(2)}/H = [4(aH)^{-4}/175] [2\Delta(\nabla^i\mathcal{R}\nabla_i\mathcal{R}) + 3\nabla^i(\Delta\mathcal{R}\nabla_i\mathcal{R})]$ in Eq. (21). The meaning of the subscript will be explained soon. Beyond the linear order, the curvature perturbation φ grows in time due to the nonlinearity in the evolution equation: $\varphi^{(2)} \propto (aH)^{-2} \propto t^{2/3}$ and $\varphi^{(3)} \propto (aH)^{-4} \propto t^{4/3}$, and it all vanishes to all orders in perturbations on superhorizon scales.

Combining Eqs. (5) and (7), we write the master differential equation to be solved for δ :

$$a^2 \left(H\dot{\delta} + \frac{3}{2}H^2\delta \right) = \frac{a^2}{4} \left(\delta R - \sigma^{ab}\sigma_{ab} + \frac{2}{3}\kappa^2 + 4HN^i\nabla_i\delta + 4H\kappa\delta \right), \quad (13)$$

and its relation to κ is given by the ADM energy constraint equation, written explicitly as

$$\begin{aligned} \frac{3}{2}H^2\delta + H\kappa + \frac{1}{a^2}\Delta\varphi &= \frac{1}{6}\kappa^2 + \frac{1}{12a^4} [(\Delta\chi)^2 - 3\nabla_i\nabla_j\chi\nabla^i\nabla^j\chi] (1 - 4\varphi) + \frac{1}{a^2} \left(4\varphi\Delta\varphi + \frac{3}{2}\nabla^i\varphi\nabla_i\varphi \right) \\ &+ \frac{1}{a^4} \left[\nabla^j\nabla^i\chi\nabla_j\varphi\nabla_i\chi - \frac{1}{3}\nabla^i\varphi\nabla_i\chi\Delta\chi \right] - \frac{3}{a^2}\varphi (3\nabla^i\varphi\nabla_i\varphi + 4\varphi\Delta\varphi). \end{aligned} \quad (14)$$

The homogeneous solution of Eq. (13) satisfies $\delta_h \propto H$, corresponding to the usual decaying mode. The particular solution that corresponds to the growing mode can be obtained as

$$\delta_p = \delta_h \int \frac{dt}{\delta_h} \frac{\text{RHS of (13)}}{a^2 H}. \quad (15)$$

We can straightforwardly compute RHS and arrange it as a sum of scale-dependent and time-dependent functions:

$$\text{RHS} \equiv \text{RHS}_1(\mathbf{x}) + \text{RHS}_2(t, \mathbf{x}) + \text{RHS}_3(t, \mathbf{x}), \quad (16)$$

where the spatial and the time dependencies of RHS_i can be further separated as $\text{RHS}_i(t, \mathbf{x}) \equiv X_i(\mathbf{x})/(aH)^{2(i-1)}$ and RHS_i vanishes at n -th order in perturbations for $n < i$.

Therefore, the particular solution in Eq. (15) is the sum of individual solutions δ_i associated with RHS_i in Eq. (15), i.e.

$\delta_p = \delta_1 + \delta_2 + \delta_3$, where $\delta_i(t, \mathbf{x}) \equiv D_i(t)X_i(\mathbf{x})$ and

$$D_i(t) = H \int \frac{dt}{(aH)^{2i}} = \frac{1}{(aH)^{2i}(i+3/2)}. \quad (17)$$

It should be emphasized that the subscript i means i -th time dependence, *not* necessarily i -th order in perturbation which is *separately* denoted by the superscript (i) . For instance, $\text{RHS}_1^{(1)} = X_1^{(1)} = -\Delta\mathcal{R}$ is a time-independent spatial function set by the initial condition, and the linear-order solution is $\delta_1^{(1)} = -D_1\Delta\mathcal{R}$, where the time-dependence of $D_1 = 2(aH)^{-2}/5 \propto t^{2/3}$ is identical to the Newtonian linear-order growth factor when normalized to unity at some epoch.

Finally, the full third-order solutions are

$$\delta_1(t, \mathbf{x}) = \frac{2}{5(aH)^2} \left[-\Delta\mathcal{R} + \frac{3}{2}\nabla^i\mathcal{R}\nabla_i\mathcal{R} + 4\mathcal{R}\Delta\mathcal{R} - 3\mathcal{R}(3\nabla^i\mathcal{R}\nabla_i\mathcal{R} + 4\mathcal{R}\Delta\mathcal{R}) \right] = \frac{\kappa_1(t, \mathbf{x})}{H}, \quad (18)$$

$$\begin{aligned} \delta_2(t, \mathbf{x}) &= \frac{2^2}{5^2(aH)^4} \left\{ \frac{1}{7}(\Delta\mathcal{R})^2 \left(5 + \frac{8}{3}\mathcal{R} \right) + \frac{2}{7}\nabla^i\nabla^j\mathcal{R}\nabla_i\nabla_j\mathcal{R}(1 - 4\mathcal{R}) + \nabla^i\mathcal{R}\Delta\nabla_i\mathcal{R}(1 - 2\mathcal{R}) - \frac{8}{7}\nabla^i\nabla^j\mathcal{R}\nabla_i\nabla_j\mathcal{R} \right. \\ &\quad + \frac{8}{21}\nabla^i\mathcal{R}\nabla_i\mathcal{R}\Delta\mathcal{R} + \left(\Delta\nabla_i\mathcal{R}\nabla^i + \frac{4}{7}\nabla_i\nabla_j\mathcal{R}\nabla^i\nabla^j - \frac{4}{21}\Delta\mathcal{R}\Delta \right) \left(D_1^{-1}\Delta^{-1}\delta_1^{(2)} + \frac{5}{2}H\Delta\chi_1^{(2)} \right) \\ &\quad \left. - \left(\frac{2 \cdot 5}{7}\Delta\mathcal{R} + \Delta\nabla_i\mathcal{R}\Delta^{-1}\nabla^i + \nabla_i\mathcal{R}\nabla^i + \frac{2 \cdot 2}{7}\nabla_i\nabla_j\mathcal{R}\Delta^{-1}\nabla^i\nabla^j \right) D_1^{-1}\delta_1^{(2)} \right\}, \end{aligned} \quad (19)$$

$$\delta_3(t, \mathbf{x}) = -\frac{1}{5 \cdot 9(aH)^2} \left[2\Delta \left(\nabla_i\mathcal{R}\Delta^{-1}\nabla^i\frac{\kappa_2^{(2)}}{H} \right) + 7\nabla^i \left(\Delta^{-1}\nabla_i\frac{\kappa_2^{(2)}}{H}\Delta\mathcal{R} \right) + 7\nabla^i \left(\delta_2^{(2)}\nabla_i\mathcal{R} \right) \right]. \quad (20)$$

By using Eq. (14), the perturbations to the extrinsic curvature are

$$\begin{aligned} \frac{\kappa_2(t, \mathbf{x})}{H} = & \frac{2^2}{5^2(aH)^4} \left\{ \frac{1}{7}(\Delta\mathcal{R})^2 \left(3 + \frac{16}{3}\mathcal{R} \right) + \frac{4}{7}\nabla^i\nabla^j\mathcal{R}\nabla_i\nabla_j\mathcal{R}(1-4\mathcal{R}) + \nabla^i\mathcal{R}\Delta\nabla_i\mathcal{R}(1-2\mathcal{R}) - \frac{16}{7}\nabla^i\nabla^j\mathcal{R}\nabla_i\mathcal{R}\nabla_j\mathcal{R} \right. \\ & + \frac{16}{21}\nabla^i\mathcal{R}\nabla_i\mathcal{R}\Delta\mathcal{R} + \left(\Delta\nabla_i\mathcal{R}\nabla^i + \frac{8}{7}\nabla_i\nabla_j\mathcal{R}\nabla^i\nabla^j - \frac{8}{21}\Delta\mathcal{R}\Delta \right) \left(D_1^{-1}\Delta^{-1}\delta_1^{(2)} + \frac{5}{2}H\Delta\chi_1^{(2)} \right) \\ & \left. - \left(\frac{2\cdot 3}{7}\Delta\mathcal{R} + \Delta\nabla_i\mathcal{R}\Delta^{-1}\nabla^i + \nabla_i\mathcal{R}\nabla^i + \frac{2\cdot 4}{7}\nabla_i\nabla_j\mathcal{R}\Delta^{-1}\nabla^i\nabla^j \right) D_1^{-1}\delta_1^{(2)} \right\}, \end{aligned} \quad (21)$$

$$\frac{\kappa_3(t, \mathbf{x})}{H} = -\frac{1}{5\cdot 3(aH)^2} \left[2\Delta \left(\nabla_i\mathcal{R}\Delta^{-1}\nabla^i\frac{\kappa_2^{(2)}}{H} \right) + \nabla^i \left(\Delta^{-1}\nabla_i\frac{\kappa_2^{(2)}}{H}\Delta\mathcal{R} \right) + \nabla^i \left(\delta_2^{(2)}\nabla_i\mathcal{R} \right) \right]. \quad (22)$$

These solutions constitute the full third-order relativistic dynamics in the proper-time hypersurface of nonrelativistic matter flows. It is well-known that $\delta_i^{(i)}$ and $\kappa_i^{(i)}$ are identical to the Newtonian solutions with the standard kernels F_i and G_i in Fourier space. The remainder $\delta_1^{(2,3)}$ and $\delta_2^{(3)}$ of the solution, and similarly for κ , represent the relativistic corrections. The second-order relativistic correction $\delta_1^{(2)}$ has been derived in literature [12, 13, 21–24], but it is the *first time* that the full third-order solution in Eqs. (18)–(22) is presented. Furthermore, our solution differs from [25, 26] in the relativistic corrections $\delta_1^{(2,3)}$ and $\delta_2^{(3)}$, clarifying the direct connection to the initial condition set by the curvature potential \mathcal{R} . In particular, the nonlinear relativistic effects in $\delta_1^{(2,3)} \propto (aH)^{-2}$ are at the heart of the recent debate in literature, and we further elaborate as follows.

In the presence of primordial non-Gaussianities, there exists a nontrivial coupling between long and short wavelength modes. For example, the local-type non-Gaussianity in the initial condition is often phrased as, up to cubic order,

$$\zeta(\mathbf{x}) = \zeta_G(\mathbf{x}) + \frac{3}{5}f_{\text{NL}}\zeta_G^2(\mathbf{x}) + \frac{9}{25}g_{\text{NL}}\zeta_G^3(\mathbf{x}), \quad (23)$$

where ζ_G is a linear-order Gaussian random field and $e^{2\zeta} \equiv 1 + 2\mathcal{R}$ in our notation convention, i.e. $\mathcal{R} = \zeta + \zeta^2 + 2\zeta^3/3$. Separating the Gaussian curvature perturbation into long and short wavelength modes $\zeta_G = \zeta_l + \zeta_s$, the curvature perturbation ζ_{short} on small scale including the non-Gaussian contributions is expressed as

$$\zeta_{\text{short}} = \zeta_s \left(1 + \frac{6}{5}f_{\text{NL}}\zeta_l + \frac{27}{25}g_{\text{NL}}\zeta_l^2 \right) + \mathcal{O}(\zeta_s^2), \quad (24)$$

where the long-short mode coupling is explicit. A similar separation of long and short wavelength modes can be performed for the matter density fluctuation in Eq. (18), but to simplify the calculation we assume that a long-wavelength mode of interest is larger than the horizon scale, neglecting its gradient:

$$\delta_{1,s}(t, \mathbf{x}) = (1 - 2\zeta_l + 2\zeta_l^2)\delta_G + \mathcal{O}(\nabla^i\zeta_s\nabla_i\zeta_s), \quad (25)$$

where $\delta_G \equiv -D_1\Delta\zeta_s$. Comparing Eqs. (24) with (25) shows that even in the absence of the primordial non-Gaussianity

$f_{\text{NL}} = g_{\text{NL}} = 0$, the nonlinear evolution of gravity in general relativity effectively generates the non-Gaussianity in the matter fluctuation $\Delta f_{\text{NL}} = -5/3$ and $\Delta g_{\text{NL}} = 50/27$. Operationally, the long-short coupling in Eq. (25) originates from the relativistic effects like $\mathcal{R}\Delta\mathcal{R}$ and $\mathcal{R}^2\Delta\mathcal{R}$ in Eq. (18) that are inherently present due to the nonlinearity of the relativistic constraint equation, even when the initial condition \mathcal{R} is Gaussian, and this calculation is the core argument that supports the nonlinear generation of non-Gaussian signatures in general relativity [12, 13, 21, 27, 28].

However, the situation is puzzling, for such effects of nonlinear gravity persist in the superhorizon limit, affecting the small-scale dynamics. This is in conflict with the equivalence principle — While long-mode fluctuations *do* affect the small-scale dynamics, their impact progressively decreases, *vanishing* in the large-scale limit, contrary to Eq. (25), where the small-scale dynamics is affected by the super-horizon wavelength mode of gravity. This situation is reminiscent of the consistency relation [14, 29, 30] in single-field inflationary scenarios, in which the bispectrum of ζ in the squeezed limit is proportional to the spectral index of the power spectrum of ζ , but is in fact identically vanishing with unobservable rescaling of spatial coordinates. We will show that the unphysical character of Eq. (25) is removed with *constant rescaling* of spatial coordinates and hence it bears no physical significance.

Noting that $\zeta_l = \zeta_l(\mathbf{x})$ is a time-independent spatial function that varies negligibly within our horizon, we consider a spatial coordinate rescaling $d\tilde{x}^i \equiv e^{\zeta_l}dx^i$, leaving the metric at early times in the *same* gauge condition

$$ds^2 = -dt^2 + a^2e^{2\zeta}dx^2 = -dt^2 + a^2e^{2\zeta_s(x)}d\tilde{x}^2, \quad (26)$$

where g_{0i} vanishes at $t \rightarrow 0$. We thus identify the curvature perturbation $\tilde{\zeta}(\tilde{\mathbf{x}})$ in the rescaled coordinate as $\tilde{\zeta}(\tilde{\mathbf{x}}) = \zeta_s(\mathbf{x})$. With the chain rule, we can find

$$\Delta\zeta(\mathbf{x}) = \tilde{g}^{ij}\nabla_i\nabla_j\zeta(\mathbf{x}) = e^{2\zeta_l}\tilde{\Delta}\zeta(\mathbf{x}), \quad (27)$$

where the Jacobian factor $e^{2\zeta_l}$ is present in addition to the rescaled Laplacian operator $\tilde{\Delta}$. Since the matter density fluctuation is a scalar, it remains unchanged under the spatial coordinate transformation $\delta(t, \mathbf{x}) = \tilde{\delta}(t, \tilde{\mathbf{x}})$. Plugging this into Eq. (25), we find

$$\delta_{1,s}(t, \mathbf{x}) = -D_1\tilde{\Delta}\tilde{\zeta}(\tilde{\mathbf{x}}) + \mathcal{O}(\tilde{\nabla}^i\tilde{\zeta}\tilde{\nabla}_i\tilde{\zeta}) \approx \delta_G, \quad (28)$$

so that $\delta_{1,s}$ is now explicitly devoid of any correlation to a long- wavelength mode fluctuation beyond our horizon. Our calculation accounting for the third-order relativistic effects is in essence equivalent to the linear-order calculation in [16], and it shares the physical basis with those in [14, 15, 31], where the second-order calculations are performed in the conformal Fermi coordinates.

Similar operations can be performed to compute $\delta_{2,s}(t, \mathbf{x})$. The quadratic terms in Eq. (19) absorb the cubic terms in proportion to $\delta_1^{(2)}$, leaving the standard Newtonian solution $\delta_2^{(2)}$. The remaining cubic terms correspond to the relativistic correction computed in [26], vanishing $\propto k^2$ in the large-scale limit, after the coordinate rescaling removes the long-mode contribution to a constant. The local observer in the proper-time hypersurface would, therefore, feel the nonlinear gravitational effect with $\tilde{\Delta}\tilde{\zeta}$. This proves that the nonvanishing correlations of long and short wavelength modes, a signature of local-type non-Gaussianity, must originate from non-gravitational forces.

Our third-order relativistic solutions in Eqs. (18)–(22) provide useful tools to analyze higher-order statistics in the proper-time hypersurface such as the bispectrum and the

trispectrum, accounting for the nonlinear relativistic effects set up by the comoving-gauge curvature perturbation at the initial epoch. It is only when the observable quantities are computed that the coordinate rescaling is naturally performed, as is the case in the computation of the single-field consistency relation.

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